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INVESTIGATION OF THE SHOCK ADIABAT OF QUASITRANSVERSE SHOCK WAVES IN A PRESTRESSED ELASTIC MEDIUM*

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Weak quasitransverse shock waves being propagated in the prestressed state are considered in an isotropic nonlinearly elastic medium. Their propagation velocity is found /1/ for such waves and the manifold of states behind the shock is investigated for a given state ahead of it (the shock adiabat). An investigation of the shock adiabat is performed below for quasitransverse waves and the location of evolutionary sections in the plane of running shear strains. The influence of initial strains is taken into account more exactly.

1. Formulation of the problem.

A nonlinearly elastic medium is given by the function

$$\Phi = \rho_0 U(\varepsilon_{ij}, S), \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial \xi_j} + \frac{\partial w_j}{\partial \xi_i} + \frac{\partial w_s}{\partial \xi_i}\frac{\partial w_s}{\partial \xi_j}\right)$$

Here U is the internal energy per unit mass of the medium, ρ_0 is the density in the unstressed state, ε_{ij} are the strain tensor components, w_i are the displacement vector components in the unstrained basis, and S is the entropy; summation over duplicate subscripts is assumed. The Lagrange coordinate system ξ_i is rectangular and Cartesian in the unstressed state. The axis ξ_3 is selected so that the surfaces of discontinuity at different times would coincide with the coordinate surfaces $\xi_3 = \text{const}$. The quantity $W = d\xi_3/dt$ characterizes the velocity of the discontinuity in the Lagrange variable ξ_3.

Only $\partial w_k/\partial \xi_3$ can undergo a discontinuity of the shock surface, the remaining $\partial w_k/\partial \xi_\alpha$, $\alpha = 1, 2$ retain their initial values and characterize the initial strain ε_{ij}^- which is considered given. It is convenient to introduce the notation $\partial w_k/\partial \xi_3 = u_k$, then $\Phi = \Phi(u_k, S, \varepsilon_{ij}^-)$. Conditions on the discontinuity

$$\left[\frac{\partial \Phi}{\partial u_k}\right] = \rho_0 W^2 [u_k], \quad [\Phi] = \frac{1}{2}\left[\frac{\partial \Phi}{\partial u_k}\right][u_k] + \left(\frac{\partial \Phi}{\partial u_k}\right)^+ [u_k] \tag{1.1}$$

are written down in /2/ (see /1/ also). Here $[a] = a^+ - a^-$ is the difference between the values of the quantity a behind and before the shock. These relationships can be used to find the states behind the shock and to determin the change in entropy S in the shock for a given velocity of the discontinuity W.

Weak intensity ($[u_k]$ small) waves being propagated in a slightly strained initial state (ε_{ij}^- small) will be considered. In this case one quasilongitudinal and two quasitransverse waves exist /1/. Here only quasitransverse waves as those studied least will be investigated. For a correct description of the nonlinear effects in quasitransverse waves, fourth order terms in the strain /2/

$$\Phi = {}^1/_2\lambda I_1^2 + \mu I_2 + \beta I_1 I_2 + \gamma I_3 + \delta I_1^3 + \xi I_2^2$$
$$\rho_0 T_0 (S - S^-) + \text{const}$$
$$I_1 = \varepsilon_{ii}, \; I_2 = \varepsilon_{ik}\varepsilon_{ik}, \; I_3 = \varepsilon_{ij}\varepsilon_{jk}\varepsilon_{ki}$$

should be taken into account in the function Φ.

It is seen from (1.1) that when studying discontinuities, only terms starting with the quadratic in $[u_k]$ are important in the expansion of Φ, and their coefficients can be considered independent of S. The notation $u_1 = u$, $u_2 = v$, $u_3 = w$, $u_1^- = U$, $u_2^- = V$ will be used later, wherein the function Φ has the form

$$\Phi = \Phi_0(S^-, \varepsilon_{ij}^-) + \frac{\partial \Phi}{\partial u_k}(S^-, \varepsilon_{ij}^-)[u_k] + \rho_0 T_0 (S - S^-) + \Psi([u_k], \varepsilon_{ij}^-) \tag{1.2}$$
$$\Psi([u_k], \varepsilon_{ij}^-) = {}^1/_2\{(\lambda + 2\mu)[w]^2 + f[u]^2 + g[v]^2\} +$$

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$$
\begin{aligned}
&2pUV\,[u]\,[v] + 2b\,[w]\,(U\,[u] + V\,[v]) + \\
&([u]^2 + [v]^2)\,(pU\,[u] + pV\,[v] + b\,[w]) + {}^1/_4\,p\,([u]^2 + [v]^2)^2 \\
&2b = \lambda + 2\mu + \beta + {}^3/_2\gamma, \quad p = {}^1/_2\lambda + \mu + \beta + {}^3/_2\gamma + \xi \\
&f = \mu + 2bI_1^- + p\,(3U^2 + V^2) - (2\mu + {}^3/_2\gamma)\,\varepsilon_{22}^- \\
&g = \mu + 2bI_1^- + p\,(U^2 + 3V^2) - (2\mu + {}^3/_2\gamma)\,\varepsilon_{11}^-
\end{aligned}
$$

Only those fourth degree terms in $\partial u_i/\partial\xi_j$ are written down for the functions Φ and Ψ which are needed for the investigation of quasitransverse waves. The expression presented for Ψ takes account of terms of a total fourth degree in $[u_i]$ and ε_{ij}^- (in contrast to /1/ where there was always a limitation to just linear terms in ε_{ij}^-).

2. Equation of the shock adiabat. The equation of the shock adiabat, i.e., the equation describing the set of states of the medium behind the discontinuity, is obtained from the sysyem (1.1) for quasitransverse shock waves in the plane uv:

$$
\begin{aligned}
&(u^2 + v^2 - R^2)\,(Uv - Vu) + 2G\,(u - U)\,(v - V) = 0 \qquad (2.1) \\
&G = (2\mu + {}^3/_2\gamma)\,(\varepsilon_{22}^- - \varepsilon_{11}^-)/\varkappa, \quad R^2 = U^2 + V^2 \\
&\varkappa = \mu + (\mu + \beta + {}^3/_2\gamma)^2/(\lambda + \mu) - 2\xi = b^2/(\lambda + \mu) - 2p
\end{aligned}
$$

The point U, V displays the initial state ahead of the shock on the uv plane. The change in the longitudinal strain component at each point of the line (2.1) is evaluated by means of the formula in which U, V, w^- enter in the form of the additive constant

$$
w = w^- + bR^2/(\lambda + \mu) - b\,(u^2 + v^2)/(\lambda + \mu)
$$

Let us note that exactly the same relation hold in simple waves /3/. The elastic properties of the material enter into (2.1) only in terms of the coefficient G which can always be made non-negative by an appropriate selection of the numbering for axes ξ_1 and ξ_2. For $G \neq 0$ it is possible to go over to new variables $\bar{u} = u/\sqrt{G}$, $\bar{v} = v/\sqrt{G}$, , whereupon the factor G in the last term in (2.1) becomes one, and the shock adiabat equation (2.1) becomes universal for all elastic media

$$
(\bar{u}^2 + \bar{v}^2 - \bar{R}^2)\,(\bar{U}\bar{v} - \bar{V}\bar{u}) + 2\,(\bar{u} - \bar{U})\,(\bar{v} - \bar{V}) = 0
$$

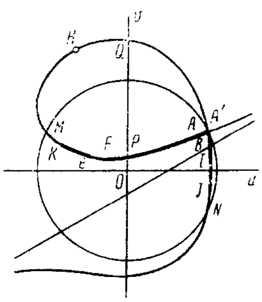

Fig.1

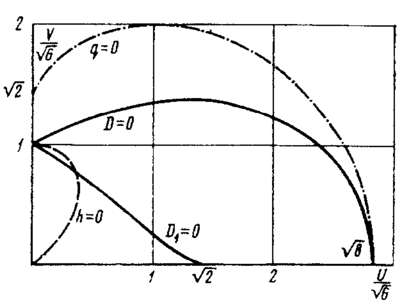

Fig.2

The form of (2.1) is such that curves with initial points U, V symmetric with respect to any of the axes u, v are symmetric to each other. Hence, it is sufficient to investigate the shock adiabat properties just for the case $U \geqslant 0, V \geqslant 0$ (Fig.1). The curve (2.1) passes through the initial point U, V and two symmetric points relative to the u, v axes: $M\,(-U, V)$ and $N\,(U, -V)$. At the initial point the curve has the re-entry point noted in Fig.1 by the two letters A and A', which is convenient for parametric representation of the equation of the curve. The tangents to the intersecting branches of the curve are mutually perpendicular, their directions yield the change in quantities in the infinitely weak or simple waves /3/:

$$
\operatorname{tg}\theta_0 = \frac{V^2 - U^2 - G \pm B}{2UV}, \quad B = [(V^2 - U^2 - G)^2 + 4U^2V^2]^{1/2} \qquad (2.2)
$$

(θ_0 and $\theta_0 + \pi/2$ are the angles between these directions at the point U, V and the u axis). The curve is a loop with tails going to infinity along the asymptote $v = V\,(u/U - 2G/R^2)$, whose direction is parallel to the radius-vector of the initial point. The slope of the asymptote to the u axis is always greater than θ_0. One of the shock adiabat tails intersects the asymptote. The ordinate v_B of the point of intersection is given by the expression

$$
v_B = V\left[1 - \frac{2U^2\,(G + R^2)}{R^4h}\right], \quad h = \frac{R^2}{G} + \frac{U^2 - V^2}{R^2}
$$

The dashes in Fig.2 depict the line $h(U, V) = 0$ in the UV plane. If the initial point U, V is in the domain where $h > 0$ (outside the curve $h = 0$ in Fig.2), then the lower branch of the tail intersects the asymptote, otherwise the upper branch intersects. When the sign of h changes, the point of intersection passes through infinity from one branch to the other.

The curve (2.1) always has three intersections with the v axis. They are the roots of the equation $v(v^2 - R^2) - 2G(v - V) = 0$, and as is seen from an investigation of this expression, are arranged as follows: $v_1 < -R$, $0 < v_2 < V$, $v_3 > R$. The point with coordinates $0, \sqrt{G}$ is a singularity for integral curves of simple waves /3/, it is always within the loop.

The points of intersection of the curve (2.1) with the u axis are determined by the equation $u(u^2 - R^2) + 2G(u - U) = 0$. It certainly has on real positive root in the range $U < u_3 < R$. . The other two negative roots will exist if the initial strains satisfy the inequality

$$q \equiv U^2G^2 - [(R^2 - 2G)/3]^3 < 0 \tag{2.3}$$

The points U, V, subject to the condition (2.3), lie outside the curve $q = 0$ depicted in Fig.2 by the dash-dot line. In this case the shock adiabat (2.1) has three intersections with the u axis: one at the tail and two in the loop. If the inequality (2.3) is not satisfied, there is just one intersection on the tail. When (2.3) becomes an equality, the loop touching the u axis at the point $u_* = -[(R^2 - 2G)/3]^{1/2} = -(UG)^{1/3}$.

3. Condition of entropy non-decrease. The shock adiabat (2.1) is the set of states into which it is possible to go from the initial stage U, V by a jump while satisfying the mass, momentum, and energy conservation laws. But only jumps with nondecreasing entropy $[S] \geqslant 0$ have the right to exist. An expression for $[S]$ has been found earlier /1/

$$8\rho_0 T_0 [S] = -\varkappa \{(u - U)^2 + (v - V)^2\} (u^2 + v^2 - R^2) \tag{3.1}$$

For $\varkappa > 0$ points within the circle

$$u^2 + v^2 = R^2 \tag{3.2}$$

while for $\varkappa < 0$ points outsie the circle correspond to entropy growth. On the circle itself $[S] = 0$. It is interesting to note that the condition $[S] = 0$ agrees with the condition $[w] = 0$. The entropy circle (3.2) intersects the shock adiabat at three points: $A(U, V)$, $M(-U, V)$, $N(U, -V)$ (Fig.1). The equation of the separating line (3.2) is independent of the material properties. The difference in the elastic properties of the medium is manifest in terms of the sign of the coefficient $\varkappa$.

A medium with $\varkappa > 0$ will be examined in detail in this paper. Hence, everywhere later, with the exception of Sect.7 where all the results will be presented that refer to media with $\varkappa < 0$, we shall consider $\varkappa > 0$.

The disposition of the shock adiabat relative to the circle (3.2) shows that there are no less than three entropy extremum points on the shock adiabat. One (the maximum) E is on the segment of the loop between A and M, another (the minimum) H is on the segment of the loop between M and A', and the third (a maximum) J is on the tail segment between A' and N. It will be shown later that there are no other extremum points.

4. Evolutionary conditions for the shock. It has been shown /1/ that not all the points of the shock adiabat that satisfy the condition $[S] \geqslant 0$ yield actually realizable shocks, since they do not all satisfy the evolutionary conditions which are needed for correctness of the boundary conditions on a discontinuity

$$\begin{gathered} c_2^- \leqslant W \leqslant c_2^+, \quad c_1^+ \leqslant W \leqslant c_3^- \\ c_1^- \leqslant W \leqslant c_1^+, \quad 0 \leqslant W \leqslant c_2^- \end{gathered} \tag{4.1}$$

Here W is the shock wave velocity, c_i^-, c_i^+ are the characteristic velocities in the states before and behind the shock, respectively. The numbering c_i is chosen that $c_1 \leqslant c_2 \leqslant c_3$, where c_3 corresponds to a quasilongitudinal wave and c_1 and c_2 to two quasitransverse waves. We call the discontinuities satisfying the upper system of two inequalities (4.1) fast while discontinuities satisfying the lower system, slow.

We find an expression for the shock velocity from conditions on the discontinuity (1.1)

$$\begin{gathered} \alpha \equiv \rho_0 W^2 = \alpha_0 - {}^1/_2\varkappa \Big\{u^2 + v^2 - R^2 + Uu + Vv + \\ \frac{G(u-U)^2 - G(v-V)^2 + 2[U(u-U) + V(v-V)]^2}{(u-U)^2 + (v-V)^2}\Big\} \\ \alpha_0 = \mu + 2bI_1^- - (\mu + {}^3/_4\gamma)(\varepsilon_{11}^- + \varepsilon_{22}^-) \end{gathered} \tag{4.2}$$

Considering the shock intensity $u - U, v - V$ to be infinitesimal, we hence find the characteristic velocities

$$\rho_0 (c_{1,2}^+)^2 = \alpha_0 - \varkappa \{u^2 + v^2 \pm {}^1/_2 [(v^2 - u^2 - G)^2 + 4u^2v^2]^{1/2}\} \qquad (4.3)$$
$$\rho_0 (c_{1,2}^-)^2 = \alpha_0 - \varkappa (R^2 \pm {}^1/_2 B)$$

It is seen from (4.3) that $c_1^- = c_2^-$ at the points $M(-U, V)$ and $N(U, -V)$ which are symmetric with the initial point and lie on the intersection of the adiabat with the entropy circle. The jump velocities at these points equal, respectively, $\rho_0 W_M^2 = \alpha_0 - \varkappa (R^2 + G)/2$ and $\rho_0 W_N^2 = \alpha_0 - \varkappa (R^2 - G)/2$. This means that for $G \neq 0$

$$c_2^\pm > W_M > c_1^\pm, \quad W_N > c_2^\pm > c_1^\pm \qquad (4.4)$$

and neither point M and N satisfies conditions (4.1).

Let us verify satisfaction of conditions (4.1) for different parts of the adiabat (2.1). The jump velocity W at each point of the adiabat is found in /1/ in the form of a function of one parameter, the polar angle θ in the system coupled to the initial point

$$\alpha(\theta) = \alpha_0 - \varkappa (R^2 - {}^1/_2 B) + \varkappa B \frac{3x(mx+n)(nx-m) - (nx-m)^2 - 2Bx^2}{(1+x^2)(nx-m)^2} \qquad (4.5)$$
$$x = \mathrm{tg}(\theta - \theta_0), \quad m = V\cos\theta_0 - U\sin\theta_0, \quad n = U\cos\theta_0 + V\sin\theta_0$$

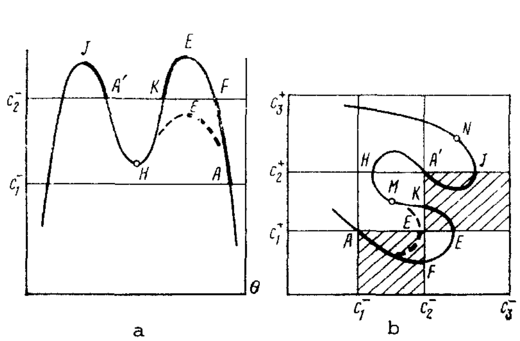

Fig.3

A graph of the function $\alpha(\theta)$ and the corresponding diagram on the c^-, c^+ plane are displayed in Fig.3 and assist in setting up the evolutionary sections on the shock adiabat. The diagram in Fig.3b is qualitative in nature, and is just to illustrate compliance with the inequalities (4.1) on different sections of the shock adiabat.

The evolutionary conditions (4.1) are satisfied in the shaded squares in Fig.3b. The section of the curve between the points A and A' in Fig.3 corresponds to a loop, the remaining parts of the curves to two tail branches. Motion along the curve from A to A' in Fig.3a corresponds to motion clockwise along the shock adiabat, and the points A and A' themselves to a double passage through the initial point U, V. The possible forms of the curves for different values of the initial strain are displayed in Fig.3 by solid and dashed lines.

The form of the function $\alpha(\theta)$ is such that its graph has no more than four intersection points upon intersecting any line $\alpha = \mathrm{const}$, and therefore, the function $\alpha(\theta)$ has not more than three extremum points. The velocity extremum points equal three since these points coincide with the entropy extremum points in conformity with /1/, and there are not less than three according to Sect.3. The velocity extremum points appear to be the Jouguet points where the velocity of discontinuity coincide with one of the characteristic velocities behind the discontinuity /1/. The intersection of a curve displaying the change in W along the adiabat and the horizontal lines $c = c_1^+$, $c = c_2^+$ corresponds to the Jouguet points in the diagram in Fig.3b. Evolutionary sections certainly about the initial point A, A' on the shock adiabat /4/. Line segments, represent W going from the points A and A' into the shaded squares correspond to them in the diagram of Fig.3b. In order to connect these diagram segments (this part of the diagram corresponds to the loop of the adiabat in Fig.1) the lines of the characteristic velocities c^+ (the two Jouguet points) must intersect twice. Then, still another Jouguet point, the point of the velocity maximum which lies between A' and N can still turn up on the rest of the curve corresponding to the two tail branches. The position of the characteristic points M and N (intersections of the adiabat with the entropy circle) is indicated in the diagram of Fig.3b in conformity with (4.4), from which it is seen that the point $W = c_2^+$ will be the Jouguet point on the tail.

5. A property of the shock adiabat at the Jouguet points. We first note one obvious consequence of equations (1.1): at the Jouguet points the shock adiabat is tangent to the integral curve giving the change in the quantities in a simple wave whose velocity agrees with the velocity of the discontinuity in the space u_k. Conversely, if the shock adiabat is tangent to the integral curve corresponding to a simple wave, then the point of tangency is a Jouguet point.

For the proof, we differentiate the first equation in (1.1) by considering the u_j^- constant

$$(\partial^2\Phi/\partial u_k \partial u_j)^+ du_j = \alpha du_k + d\alpha\,[u_k], \quad \alpha = \rho_0 W^2$$

At the Jouguet point α reaches the extremum /1/ and $d\alpha = 0$. The remaining terms show that the changes du_k agree with the changes in a simple wave being propagated at the velocity $W = \sqrt{\alpha/\rho_0}$. On the other hand, if the shock adiabat is tangent to the integral curve corresponding to a simple wave, then this means that the vector du_j along the shock adiabat is parallel to the eigenvector of the matrix $\partial^2\Phi/\partial u_k \partial u_j$. The increment $d\alpha \neq 0$ only upon satisfaction of the additional condition that the vector $[u_k]$ is parallel to the vector du_k at the point under consideration. This condition is not satisfied at any point of the shock adiabat under consideration, as follows from writing its equation in polar coordinates with respect to the initial point /1/. Therefore, $d\alpha = 0$ and the point of tangency is a Jouguet point.

6. **Location of evolutionary sections on the shock adiabat.** We shall consider $G > 0$, the case $G = 0$ is not considered here. For $G > 0$ a part of the loop from the initial point $A(U, V)$ to the point $M(-U, V)$ drops within the entropy circle, as does a part of the tail from the initial point $A'(U, V)$ to the point $N(U, -V)$ (Fig.1). Two evolutionary sections, one on the loop and one on the tail, adjoin the initial point. The segment adjoining the initial point A' is terminated by the Jouguet point J, where $W = c_2^+$. It is seen from the diagram in Fig.3b that there can be one or two evolutionary sections on the loop. This depends on the quantity of roots for the function $W(\theta) - c_2^-$, which, in conformity with (4.5), agrees with the number of roots of the cubic equation

$$F(x) = 3mnx^3 + (2n^2 - 3m^2 - 2B)x^2 - mnx - m^2 = 0 \qquad (6.1)$$

It certainly has one positive root yielding the point $W = c_2^-$ on the tail of the adiabat outside the entropy circle (Fig.3). Moreover, if the quantity D defined by the expression

$$D \equiv \bar{B}(\bar{R}, \omega)\bar{R}^2(\bar{R}^4 - 6\bar{R}^2\omega + 11) - \bar{R}^8 + 5\bar{R}^6\omega + 6\bar{R}^4\omega^2 - 13\bar{R}^4 + 3\bar{R}^2\omega + 16 \qquad (6.2)$$

$$\bar{R}^2 = R^2/G, \quad \omega = (U^2 - V^2)/R^2 \quad (-1 \leqslant \omega \leqslant 1)$$

$$\bar{B}(\bar{R}, \omega) = B/G = (\bar{R}^4 + 2\bar{R}^2\omega + 1)^{1/2}$$

is negative, then there are still two negative roots for the polynomial (6.1), to which two points, where $W = c_2^-$, lying on the loop, correspond on the shock adiabat. In this case two evolutionary sections will be on the loop. One segment will start at the point A and terminate at the point F, where $W = c_2^-$, while the other starts at the Jouguet point E, where $W = c_1^+$ and terminates at the point K where $W = c_2^-$. The graphs corresponding to this case are displayed by solid lines in Fig.3. If $D > 0$ then there are not negative roots in the polynomial (6.1) and there is one evolutionary section on the loop, starting at the point A and terminating at the Jouguet point E where $W = c_1^+$. This case is shown by dashes in Fig.3. The disappearance of D corresponds to the presence of a multiple root in the polynomial (6.1) and to merger of the evolutionary sections on the loop.

The equation $D = 0$ defines the curve displayed by a solid line in Fig.2 and constructed from points found numerically, on the plane $U/\sqrt{G}$, $V/\sqrt{G}$. Outside the curve $D < 0$ and inside $D > 0$.

We now find the position of the ends of the evolutionary sections in the plane uv. We first find the position of the entropy extremums that are simultaneously Jouguet points and points of velocity extremums. According to Sect.4 there are three such points: E, H, J (Fig. 1). A simple but tedious calculation of the derivative of the entropy along the shock adiabat shows that intersections of the shock adiabat with the axes v and u (Fig.1) are satisfied at the points P, Q, I by the inequalities

$$\left(\frac{dS}{du}\right)_P < 0, \quad \left(\frac{dS}{du}\right)_Q > 0, \quad \left(\frac{dS}{dv}\right)_I < 0$$

Because there are three entropy extremums on the shock adjabat, it follows from the inequalities presented that none lies in the first quadrant. Hence, it is evident that the point H lies in the second quadrant, and the point J in the fourth.

If the loop segment enters the third quadrant, then the point E where $W = c_1^+$ lies on it.

Indeed, it follows from the symmetry of the problem that a solution of the simple wave type exists at which $v = 0$. This simple wave corresponds to the characteristic velocity c_1 (this becomes evident if the initial point A of the shock adiabat is placed on the axis and low amplitude discontinuities are examined). In the case under consideration the axis u, representing an integral curve corresponding to the simple wave, intersects the loop of the shock adiabat. Because the field of integral curves corresponding to simple waves has no singularity within the third quadrant (these singularities lie on the v axis at the points $v=\pm\sqrt{G}$) on the loop segment of the shock adiabat in the third quadrant, there should be a point of tangency with the integral curve corresponding to c_1. From what has been said in the previous section, it follows that this point in the third quadrant is the Jouguet point $W=c_1^+$ (the point E).

If the shock adiabat is not enter into the third quadrant, then the point E lies in the second quadrant. If the shock adiabat is tangent to the u axis, then the point of tangency is the Jouguet point E.

The points F and K at which $W=c_2^-$ always lie in the second quadrant if they are on the shock adiabat.

To show this we consider the shock adiabat passing through some of these points as through the initial point. Because of symmetry of (1.1) with respect to the states before and behind the discontinuity, this shock adiabat passes through the point A. Hence, the Jouguet condition $W=c_2(A)$ will be satisfied at the point A. Therefore, an entropy extremum should be at this point on the shock adiabat under consideration, which can be only a minimum since $S(F)>S(A)$, $S(K)>S(A)$. However, it has been shown above that the point where the minimum entropy is achieved always lies on the quadrant adjacent to the initial point. Therefore, the points F and K are in the second quadrant.

Therefore, one evolutionary segment of the shock adiabat corresponding to fast waves always starts at the initial point A ($u=U>0$, $v=V>0$) and terminates at the Jouguet point J lying in the fourth quadrant within the entropy circle.

If the initial point U, V lies in the domain $D<0$ (outside the solid curve in Fig.2), then there are two evolutionary segments on the loop. One corresponding to slow waves starts at the point A and terminates at the point F in which $W=c_2^-$ which is always in the second quadrant. The second, corresponding to fast waves, starts at the point E and terminates at the point K. The point K at which $W=c_2^-$ always lies in the second quadrant. The point E at which $W=c_1^+$ lies in the second or third quadrant, depending on whether the loop enters the third quadrant. This latter will occur if the initial point lies in the domain $q<0$ (outside the dot-dash curve in Fig.2).

If the initial point lies in the domain $D\geqslant 0$, then there is one evolutionary section on the loop which is terminated by the Jouguet point E. Evolutionary sections of the adiabat for media with $\varkappa>0$ are displayed by heavy lines in Figs.1 and 3.

Let us note that the evolutionary conditions (4.1) turn out to be stronger for the medium under consideration than the condition of nondecreasing entropy. In particular, purely transverse discontinuities with $[w]=0$ are not evolutionary (the points M and N on the adiabat, for which $[S]=0$, correspond to these discontinuities).

7. Media with $\varkappa<0$. By changing the numbering of the axes, it is again possible to make $G>0$. Then the equation of the shock adiabat (2.1), and therefore, also its form on the plane uv (Fig.1) are completely conserved. There remain the expressions (4.2), (4.5) for the shock velocity W. Only in the formulas for the characteristic velocities (4.3) must the numbering for $c_{1,2}$ be replaced because of the change in sign of $\varkappa$ so that as before there would be $c_1<c_2$. The equation of the circle on which $[S]=0$ indeed retains its form. But the domains where S is less and where S is greater than in the initial state change places. For $\varkappa<0$ those sections of the shock adiabat which turned out to be outside the entropy circle, part of the loop and two tails, can be realizable from the viewpoint of nondecreasing entropy.

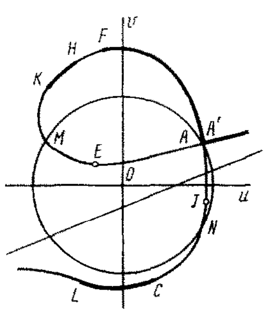

Fig.4

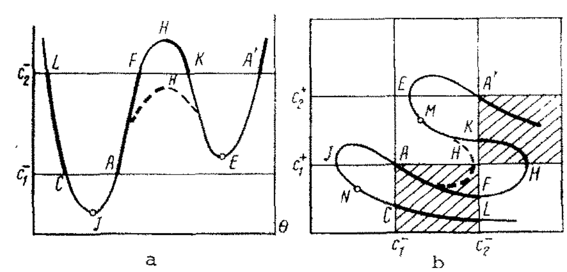

Fig.5

It is easy to find the position of the Jouguet points where $W = c_{1,2}^+$ on the adiabat. These are the extremum points of the functions W and S. They have already been found for $\varkappa > 0$. And since $\varkappa$ only enters as a factor in the variable part of the functions W and S (additions to the main constant quantity because of the nonlinearity), then the extremum points remain as before, only the maximums are replaced by minimums, and vice versa. There will be three extremum points in all (Jouguet points): one maximum and two minimum points. The maximum point H lies on the loop outsie the entropy circle in the second quadrant above the point $M(-U, V)$ (Fig.4). One of the minimum points J lies within the entropy circle on the tail in the fourth quadrant above the point $N(U, -V)$. The other minimum point E of S and W is on the lower branch of the loop (within the S-circle) and is in the third quadrant if the loop intersects the u axis or in the second below the point M if the loop does not intersect the abscissa axis.

A graph of the change in the jump velocity W as a function of the polar angle θ is represented in Fig.5a. It is seen there where the evolutionary sections should be located. They are marked off by heavy lines. To be graphic, a diagram of the change in W on the c^-, c^+ plane (Fig.5b) is also presented, which will aid in determining the quantity and position of the evolutionary sections. As before, the different possible forms of the curve as a function of the initial strain U, V, G are displayed by continuous and dashed lines in Fig.5. It is seen at once from the diagram in Fig.5b that $W = c_1^+$ at the point H of the maximum, and we have $W = c_2^+$ on the lower branch of the loop (the point E) and $W = c_1^+$ on the tail (the point J) at the minimum points. One of the evolutionary sections is the whole upper tail, starting from the point A'. It is entirely in the first quadrant (Fig.4).

Still another evolutionary section CL is on the other branch of the tail, where is emerges beyond the limits of the entropy circle. An investigation analogous to Sect.6 shows that this section starts in the fourth quadrant and terminates in the third. Still other evolutionary sections will be on the outer part of the loop. There can be one or two depending on how many roots are in $W - c_2^-$. In conformity with (4.5), they will be the roots of the equation

$$F_1(x) = n^2x^3 + mnx^2 + (3n^2 - 2m^2 - 2B)x - 3mn = 0 \qquad (7.1)$$

There can be one or three (Fig.5) depending on the sign of the discriminant D_1 of (7.1)

$$D_1 = \bar{B}\bar{R}^2(\bar{R}^4 - 6\bar{R}^2\omega + 11) + \bar{R}^8 - 5\bar{R}^6\omega - 6\bar{R}^4\omega^2 + 13\bar{R}^4 - 3\bar{R}^2\omega - 16$$

For $D_1 \geqslant 0$ equation (7.1) has just one root and its corresponding point is the end of the single evolutionary section on the adiabat loop. For $D_1 < 0$ there are three points F, H, K (Fig.4) on the loop, where $W = c_2^-$. They are all in the second quadrant. The domain $D_1 < 0$ is dispalyed in the plane of the initial data in Fig.2 within the curve $D_1 = 0$ which has been found numerically.

The evolutionary sections of the adiabat for media with $\varkappa < 0$ are denoted by heavy lines in Figs.4 and 5.

8. Incompressible media.

Let us note that incompressible, isotropic media enter, in particular, into isotropic media. The medium becomes incompressible if the quantity $\chi\Delta^2$ is appended to any previously chosen expression for Φ, where Δ is the volume expansion coefficient

$$\Delta = (1 + 2I_1 + 2I_1^2 - 2I_2 + {}^4/_3I_1^3 - 4I_1I_2 + {}^8/_3I_3)^{1/2} - 1$$

and the quantity χ then tends to infinity. The addition of the quantity mentioned results in a change in the coefficients in the expression for Φ such that there will be $\lambda + 2\chi$ in place of λ, $\beta - 2\chi$ in place of β, $\delta + \chi$ in place of δ, ξ χ in place of ξ. The remaining coefficients as well as b, p, f, g do not change in the expression for Ψ. Therefore, the passage to incompressible media can be performed if the quantity λ tends to infinity in the expressions for Ψ and all the subsequent formulas while the coefficients $\mu, \beta, \gamma, b, p, f, g$ are unchanged. This results in $\varkappa = -2p$ and $w = w^-$ being set into the formulas presented above for incompressible media. All the deductions will correspondingly be as before.

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